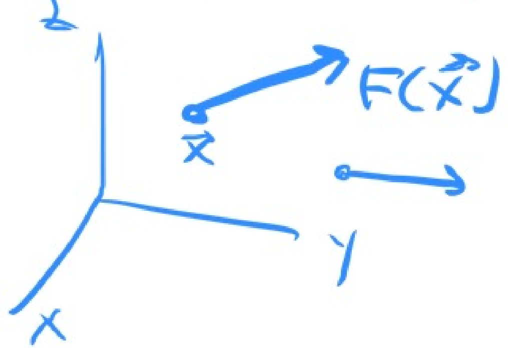
Section 4.3 Vector Fields

<u>Def.</u> A vector field on a region $A \subset \mathbb{R}^n$ (usually $n=2,3$) is a map

$$F: A \to \mathbb{R}^n$$

which assigns to each $\vec{x}$ in A a vector $F(\vec{x})$



vector fields occur in many areas in physics and engineering

Examples

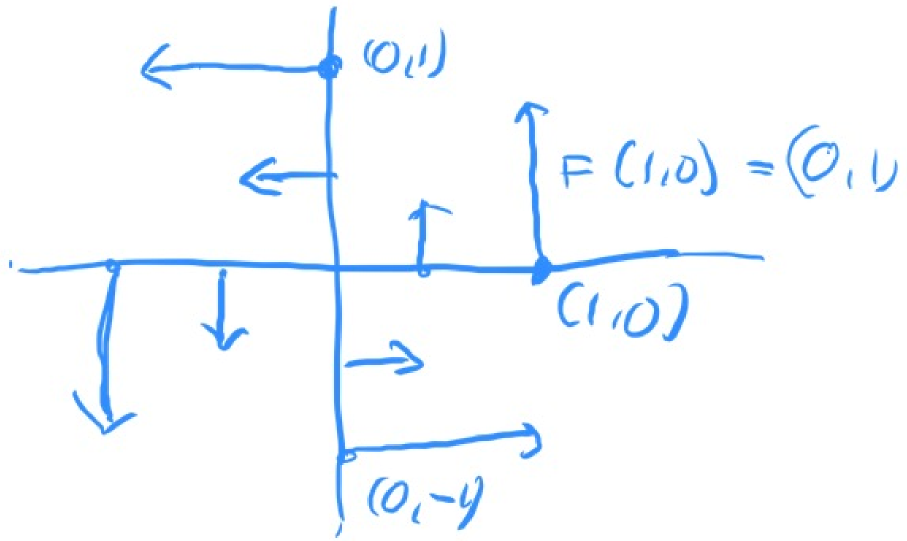
(1) describe flows of liquids

$\vec{x}$ position

$F(\vec{x})$ velocity of particle at position $\vec{x}$

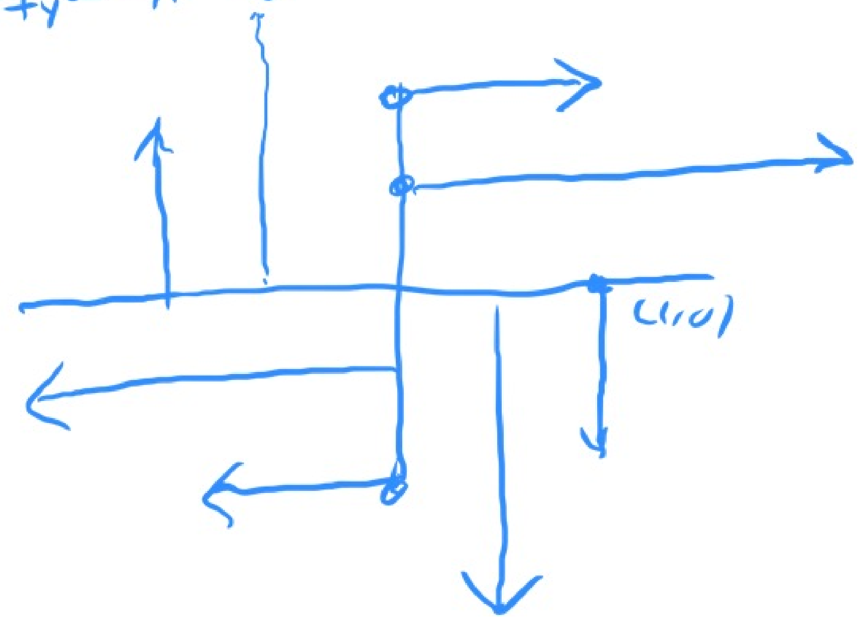
(2) rotation of a turntable described by vector field $F(x,y) = (-y, x)$

$F(x,y) = (-y, x)$



~ anti clockwise rotation

③ $F(x,y) = \frac{1}{x^2+y^2}(y, -x)$



here:
clockwise rotation
velocity increases when approaching (0,0)

~ models water at a drain

4 Gravitational Force Field

$$\boxed{F = - \frac{mMG}{r^3} \vec{r}}$$

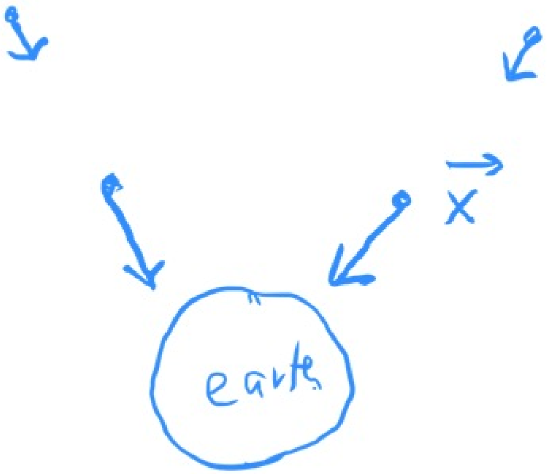
where

M mass of earth

G gravitational const

m mass of particle

$\vec{r} = (x, y, z)$ (0,0,0) center of earth



$r = \|\vec{r}\| = \sqrt{x^2 + y^2 + z^2}$

$r \geq$ radius of earth

5 Gradient Fields

$f: \mathbb{R}^3 \to \mathbb{R}$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

assigns to point (x,y,z) the vector $\nabla f(x,y,z)$

example: $f(x,y,z) = x^2 y + y z^2$

has gradient field $\nabla f(x,y,z) = (2xy, x^2 + z^2, 2yz)$

<u>Question</u> Are all vector fields gradient fields?

try $F(x,y) = (-y, x)$ gradient field?

<u>method 1</u> assume $F = \nabla f \Rightarrow \frac{\partial f}{\partial x} = -y$

integrate: $f(x,y) = \int -y \, dx$

$= -yx + C(y)$

$\Rightarrow x = \frac{\partial f}{\partial y} = \partial/\partial y \, (-yx + C(y)) = -x + C'(y)$

$x = -x + C'(y)$ ↯ not possible

method 2: assume $F = \nabla f$

$$\frac{\partial f}{\partial x} = -y \qquad \frac{\partial f}{\partial y} = x$$

calculate $\frac{\partial^2 f}{\partial x \partial y}$ in two different ways

$$\frac{\partial^2 f}{\partial x \partial y} = f_{xy}$$

ⓐ $f_{yx} - \partial/\partial y \left(\frac{\partial f}{\partial x}\right)$

$= \partial/\partial y\,(-y) = -1$

ⓑ $f_{xy} = \partial/\partial x \left(\frac{\partial f}{\partial y}\right)$

$= \partial/\partial x\,(x) = 1$

$\neq$ ↯

contradiction

Flow Lines

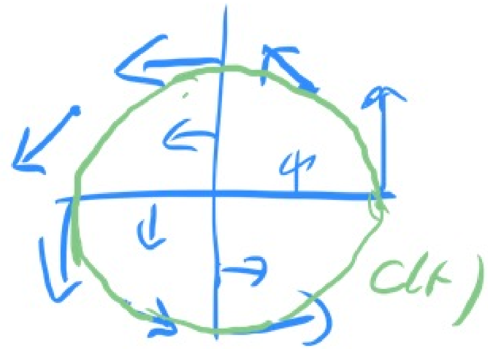
Def. A flow line of the vector field F is a curve $c(t)$ such that

tangent vector at $c(t)$ = vector of F at $c(t)$

i.e. $$c'(t) = F(c(t))$$

example: $F(x,y) = (-y, x)$

check: $c(t) = (\cos t, \sin t)$ is a flow line!



$F(x,y) = (-y, x) \qquad c(t) = (\cos t, \sin t)$

$c'(t) = (-\sin t, \cos t)$

$$F(c(t)) = F(\underset{\underset{x}{\shortparallel}}{\cos t}, \underset{\underset{y}{\shortparallel}}{\sin t}) = (-\sin t, \cos t)$$

$\Rightarrow$ flow line!

<u>Remark</u>: Calculating flow lines usually difficult
need to solve differential equations!

<u>Question:</u> Determine all possible flow lines
for $F(x,y) = (-y, x)$!

$\boxed{F(x,y) = (-y, x)}$

Solution: Let $c(t) = (x(t), y(t))$

$$\Rightarrow \quad c'(t) = (x'(t), y'(t))$$

$$F(c(t)) = F(x(t), y(t)) = (-y(t), x(t))$$

compare coordinates in $c'(t) = F(c(t))$

$$\Rightarrow \quad x'(t) = -y(t) \xrightarrow{\text{differentiate}} x''(t) = -y'(t)$$

$$y'(t) = x(t)$$

$$\Rightarrow \quad x''(t) = -x(t)$$

general solution: $x(t) = c_1 \cos t + c_2 \sin t$

$$y(t) = -x'(t) = c_1 \sin t - c_2 \cos t$$

$$\Rightarrow c(t) = (x(t), y(t)) = (c_1 \cos t + c_2 \sin t,\ c_1 \sin t - c_2 \cos t) = \boxed{c_1 (\cos t, \sin t) + c_2 (\sin t, -\cos t)}$$